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A NUMERICAL METHOD TO DETERMINE THE OPTIMAL STOPPING BOUNDARY FOR INSTALLMENT OPTION

KARIM IVAZ[1], ALI BEIRANVAND[1]

Abstract. In this paper we consider the European continuous installment call option on foreign currency exchange rate as underlying asset. Using the Black-Scholes model as the underlying asset model and applying arbitrage pricing theory, we get the parabolic partial differential equation governing the value of installment option. Then, to determine the location of the stopping boundary and the value of the European installment option, the front fixing method will be applied.

Keywords: installment option, Black-Scholes model, foreign currency, front fixing method, free boundary problem.

AMS Subject Classification: 91Gxx, 91G20, 91G20, 91G80, 35R35.

1. Introduction

Since the papers by Black-Schloes [8] and Merton [29], financial mathematics became one of the most important field of research. From that time many financial instruments have been introduced to financial industry and markets. One of these instrument is installments option.

Installment option is a type of security in which instead of paying lump sum up front, the premium is paid over the life of the option. Thus the holder has the right to terminate the contract at any time prior to maturity. Based on the net present value of contract the holder makes decision to continue or stop the payments. Hence the owner will continue to pay future installments if the option worth the net present value of the remained payments. Otherwise the holder allows the contract to laps. After the last installment, the contract will become a vanilla option.

Installment options are traded in foreign currency market between banks and corporations. Several contracts can be considered as installment options such as: some life insurance contracts, capital investment projects, installment warrant, some contracts in pharmacy and employee stock options [7, 18, 20, 27, 28, 31], respectively.

There are a lot of methods to price option contracts. Now, we briefly introduce some of the most important methods. Boyle [9] and Boyle et al. [10] used Monte Carlo method for pricing options. Brenann and Shwarz introduced finite difference method for valuing American option [11]. For solving option pricing problem, Cox, Ross and Rubinstein applied lattice method [17]. Penalty and front fixing methods were used by Nielsen et al. [30] for pricing American option in continuous time. Zvan et al. used penalty method to price American option under stochastic volatility model in discrete time [34]. Finite element and finite volume methods were applied by Allegretto et al. [1] and Wang et al. [32, 33] to value option contracts, respectively. Geske and Johnson solved American put option by Richardson extrapolation [21]. For pricing American

[1]Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
e-mail: ivaz@tabrizu.ac.ir.
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option, Baron-Adesi and Whaly applied an analytic approximation method [5]. Analytical method of lines was used by Carr and Faguet [14] to value American option problem. Huang et al. introduced integral equation approach for pricing and hedging American option [23]. Broadi and Detemple used capped option approximation for valuing American options [12].

In the case of continuous installment options a few work exists. Ciurlia and Roko applied the multi piece exponential function (MEF) to solve the free boundary problem arisen from the American continuous installment option [16]. European and American continuous-installment options are investigated using Laplace-Carson transform by Kimura in [25] and [26], respectively. In [15] Ciurlia priced European continuous installment options using Monte Carlo approach. Alobaidi used the integral transform to price European continuous installment options [3]. She also analyzed the behavior of the price of European installment options near expiry [2]. Perpetual American continuous installment option was studied by Ciurlia and Caperdoni [13].

Setting foreign currency exchange rate as underlying asset, we consider the installment option. In this paper, a front fixing method for solving free boundary problems will be introduced for valuing European installment option. In this technique, the fee boundary problem is transformed to a fixed boundary problem, in which allows us to discretize the problem by the use of finite difference method.

The rest of the paper is organized as follows: Section 2 presents the modeling of European continuous installment option under Black-Scholes model. In section 3 the front fixing method is applied to solve the nonlinear parabolic partial differential governing the European continuous installment call option. In Section 4, finite difference is applied to approximate the problem. In Section 5, numerical results for the price of European continuous installment call option are computed and the graphs of the stopping boundary are depicted.

2. The model

In this section, we consider foreign currency exchange rate under Black-Scholes model [8, 29] and model installment option on this asset. The Black-Scholes model assumes that the risk-neutral process of the underlying asset price evolves according to the stochastic differential equation (SDE):

$$dS_t = (r_d - r_f)S_t dt + \sigma S_t dW_t, \tag{1}$$

where r_d is the domestic risk-free interest rate, r_f is the foreign risk-free interest rate, σ is volatility and W_t is the Wiener process. Suppose that $V(S_t, t; q)$ is the price of a European continuous-installment option. By Ito's lemma, we have

$$dV_t = (\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + (r_d - r_f)S_t \frac{\partial V_t}{\partial S_t} - q)dt + \sigma S_t \frac{\partial V_t}{\partial S_t} dW_t, \tag{2}$$

where q is the rate of installment paid by the option holder per unit time. Next, we set up a portfolio consisting of the continuous-installment option and $-\Delta$ units of the underlying asset. Define Π_t as the value of the portfolio, i.e,

$$\Pi_t = V_t - \Delta S_t. \tag{3}$$

The change in the value of this portfolio in a small time interval is given by

$$d\Pi_t = dV_t - \Delta dS_t - \Delta(S_t r_f dt). \tag{4}$$

Substituting (1) and (2) into (4) yields

$$d\Pi_t = (\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} + (r_d - r_f)S(\frac{\partial V_t}{\partial S} - \Delta) - q - \Delta S_t r_f)dt + \sigma S_t(\frac{\partial V_t}{\partial S} - \Delta)dW_t.$$

To make the portfolio riskless, we choose $\frac{\partial V_t}{\partial S} = \Delta$. Then,

$$d\Pi_t = (\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - S_t r_f \frac{\partial V_t}{\partial S} - q)dt. \tag{5}$$

On the other hand, in the absence of arbitrage opportunities, this riskless portfolio must earn the return r_d,

$$d\Pi_t = r_d \Pi_t dt. \tag{6}$$

Plugging from (3) and (5) into (6), we obtain

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} + (r_d - r_f)S\frac{\partial V_t}{\partial S} - r_d V_t = q.$$

Noting that q is the rate of installment, it must be positive. For $q = 0$, the above parabolic partial differential equation becomes the equation corresponding to European vanilla options.

3. Front fixing method

Let $c(S_t, t; q)$ be the value of the European installment call option with the maturity T, the exercise price K and the payoff function $\max(S_T - K, 0)$. In this case an optimal stopping problem arises because of the opportunity to terminate the contract at any time $t \in [0, T]$. Hence, one should find such points (S_t, t) that optimally terminates the contract. The value of call option can be computed as the solution of the following optimal time stopping problem[25]

$$c(S_t, t; q) = ess\ sup_{\tau \in [t,T]} E[\chi_{\{\tau \geq T\}} e^{-r_d(T-t)} max(S_T - K, 0) - \frac{q}{r_d}(1 - e^{-r(\tau \wedge T - t)})|\mathcal{F}_t],$$

where $\tau \wedge T = \min(\tau, T)$ and $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_t, \mathbf{P})$ is a filtered probability space and τ is a stopping time of its filtration. The time at which the above relation gets its supremum is called an optimal stopping time $\tau \in [0, T]$. The domain of definition is $\mathcal{D} = [0, T] \times [0, \infty)$. Let us denote the stopping region and the continuation region by $\mathcal{S}$ and $\mathcal{C}$, respectively. Then, the stopping region is

$$\mathcal{S} = \{(S_t, t) \in \mathcal{D} \mid c(S_t, t; q) = 0\}.$$

The optimal stopping time τ^* is characterized by

$$\tau^* = inf\{\tau \in [t, T] | (S_\tau, \tau) \in \mathcal{S}\}.$$

Since the continuation region is the complement of the stopping region in $\mathcal{D}$, it is given by

$$\mathcal{C} = \{(S_t, t) \in \mathcal{D} \mid c(S_t, t; q) > 0\}.$$

The boundary at which the regions $\mathcal{S}$ and $\mathcal{C}$ separated from each other is called stopping or free boundary

$$S_f(t) = \inf\{S_t \in [0, \infty) \mid c(S_t, t; q) > 0\},\ t \in [0, T].$$

The valuation of European call can be done through the solution of the following inhomogeneous partial differential equation [25]

$$\frac{\partial c}{\partial t}+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 c}{\partial S^2}+(r_d-r_f)S\frac{\partial c}{\partial S}-r_d c=q, \tag{7}$$

subject to the terminal condition

$$c(S_T,T;q)=\max(S_T-K,0), \tag{8}$$

and along with boundary conditions

$$\lim_{S_t\to S_f(t)} c(S_t,t;q)=0,\ \lim_{S_t\to S_f(t)}\frac{\partial c}{\partial S}=0,\ \lim_{S_t\to\infty}\frac{\partial c}{\partial S}<\infty. \tag{9}$$

On the free boundary S_f, there are two conditions $\lim_{S_t\to S_f(t)} c(S_t,t;q)=0, \lim_{S_t\to S_f(t)}\frac{\partial c}{\partial S}=0$ called value matching and smooth pasting conditions, respectively. They show the values of portfolio and its derivative with respect to S, that is, its delta. These conditions imply that the initial premium and the slope are continuous across the free boundary.

In this problem there are two unknowns $c(S_t,t;q)$ and $S_f(t)$. This problem is called free boundary problem. Solving this type of problem is a challenging work. In this paper, for the first time, we will apply front fixing method to solve the mentioned free boundary problem. As seen in formulation (7)-(9), the free or stopping boundary belongs to the domain of the definition of the problem. Therefore difficulty arises in solving such a problem. In this paper, we will apply front fixing method [30] to solve the above problem. Nielsen et al. used this method to solve American put option [30]. The basic idea of front fixing method is to remove the free boundary from the domain and to add it to the partial differential equation (PDE) (7). The resulted problem is a nonlinear PDE with known and fixed boundary conditions. Now consider the following change of variables

$$x=\frac{S}{KS_f(t)},\ u(x,t;q)=c(S,t;q), \tag{10}$$

It is simply seen that this transforms the domain $S\in[S_f(t),\infty)$ to the domain $x\in[E,\infty)$ where $E=\frac{1}{K}$. Now, we want to reformulate the PDE (7), terminal condition (8) and boundary conditions (9) in terms of (x,t). By chain rule differentiation, we have

$$\frac{\partial c}{\partial S}=\frac{\partial u}{\partial x}\frac{dx}{dS}=\frac{1}{KS_f(t)}\frac{\partial u}{\partial x}, \tag{11}$$

$$\frac{\partial^2 c}{\partial S^2}=\frac{\partial}{\partial S}(\frac{\partial c}{\partial S})=\frac{1}{K^2S_f^2(t)}\frac{\partial^2 u}{\partial x^2}. \tag{12}$$

Also differentiating u with respect to t yields

$$\frac{\partial u}{\partial t}=\frac{\partial c}{\partial t}+\frac{\partial c}{\partial S}\frac{dS}{dt},$$

Substituting from (11) and considering $\frac{dS}{dt}=KxS_f'(t)$, one can get

$$\frac{\partial c}{\partial t}=\frac{\partial u}{\partial t}-x\frac{S_f'}{S_f}\frac{\partial u}{\partial x}. \tag{13}$$

Substituting from (11), (12) and (13) into (7) yields

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}+(r_d-r_f-\frac{S_f'(t)}{S_f(t)})x\frac{\partial u}{\partial x}-r_d u=q.$$

Clearly, this is a nonlinear partial differential equation. At this time, we reformulate terminal and boundary conditions. For terminal condition (8), we have

$$u(x,T;q) = c(S,T;q) = \max(S_T - K, 0) = \max(KS_f(T)x - K, 0).$$

In [25], it is proved that $S_f(T) = K$. Using this fact and $x \geq 1$, one can get

$$u(x,T;q) = K(Kx-1).$$

Since $S = S_f(t)$ is equivalent to $x = E$ under the change of variables (10), for boundary conditions (9), the following relations hold

$$\begin{aligned} &u(E,t;q) = c(S_f(t),t;q) = 0,\\ &\frac{\partial u}{\partial x}(E,t;q) = KS_f(t)\frac{\partial c}{\partial S}(S_f(t),t;q) = 0,\\ &\lim_{x\to\infty}\frac{\partial u}{\partial x} = KS_f(t)\lim_{S\to\infty}\frac{\partial c}{\partial S} < \infty. \end{aligned}$$

Therefore the reformulation of the problem (7)-(9) is a nonlinear partial differential equation with terminal and boundary conditions

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + (r_d - r_f - \frac{S_f'(t)}{S_f(t)})x\frac{\partial u}{\partial x} - r_d u = q, \tag{14}$$

$$u(x,T;q) = K(Kx-1), \tag{15}$$

$$u(E,t;q) = 0, \tag{16}$$

$$\frac{\partial u}{\partial x}(E,t;q) = 0, \tag{17}$$

$$\lim_{x\to\infty}\frac{\partial u}{\partial x} < \infty. \tag{18}$$

Comparisons of the boundary conditions in this problem and that of (7)-(9) reveals that the boundary conditions in this problem does not contain the free or stopping boundary $S_f(t)$ and this term is translated to nonlinear PDE (14). Solving the above problem is equivalent to find $u(x,t)$ and $S_f(t)$.

4. Finite difference approximation

In continuation, the finite difference method will be applied to solve the above problem. In this method, the derivatives in the PDE (14) are approximated by difference schemes. To apply finite difference method to the above problem, we need to bound the domain of the mentioned problem. Let x_∞ be sufficiently large number. This value plays the role of ∞ in the problem (14)-(18). For the boundary condition (18), the Neumann boundary conditions at x_∞ is defined by

$$\frac{\partial u}{\partial x}(x_\infty,t) = \frac{\partial}{\partial x}(K(Kx-1))|_{x=x_\infty} = K^2.$$

Let $M, N > 0$ be integer numbers and define

$$\begin{aligned} &h = \frac{x_\infty - E}{M+1},\ k = \frac{T}{N+1},\\ &x_i = E + ih,\ i = 0,1,\cdots,M+1,\\ &t_j = jk,\ j = 0,1,\cdots,N+1. \end{aligned}$$

Also assume that

$$u_i^j \approx u(x_i, t_j),\ 0 \le i \le M+1,\ 0 \le j \le N+1,$$
$$S_f^j \approx S_f(t_j),\ 0 \le j \le N+1.$$

At this moment, the partial derivatives in (14) will be approximated by difference schemes. At first, for time derivatives, we apply the schemes

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u_i^{j+1} - u_i^j}{k},\ 0 \le i \le M+1,\ 0 \le j \le N,$$
$$S_f'(t_j) \approx \frac{S_f^{j+1} - S_f^j}{k},\ 0 \le j \le N.$$

Also for partial derivatives in x, we have

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2},\ 0 \le i \le M+1,\ 0 \le j \le N,$$
$$\frac{\partial u}{\partial x}(x_i, t_j) \approx \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{h^2},\ 0 \le i \le M+1,\ 0 \le j \le N.$$

Substituting these relations in (14) yields

$$\frac{u_i^{j+1} - u_i^j}{k} + \frac{1}{2}\sigma^2 x_i^2 \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}$$
$$+ (r_d - r_f - \frac{S_f^{j+1} - S_f^j}{kS_f^{j+1}}) x_i \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} - r_d u_i^j = q.$$

This scheme is called explicit finite difference scheme for equation (14). A simple calculation, for $0 \le i \le M+1$, yields

$$u_i^j - \lambda_i^{j+1} S_f^j = \alpha_i u_{i-1}^{j+1} + \beta_i u_i^{j+1} + \gamma_i u_{i+1}^{j+1} - kq,\ j = N, N-1, \cdots, 0, \tag{19}$$

where

$$\lambda_i^{j+1} = \frac{x_i}{2hS_f^{j+1}}(u_{i+1}^{j+1} - u_{i-1}^{j+1}),$$

$$\alpha_i = \frac{1}{2}\sigma^2 x_i^2 \frac{k}{h^2} - x_i(r_d - r_f - \frac{1}{k})\frac{k}{2h}, \tag{20}$$
$$\beta_i = 1 - \sigma^2 x_i^2 \frac{k}{h^2} - kr_d, \tag{21}$$
$$\gamma_i = \frac{1}{2}\sigma^2 x_i^2 \frac{k}{h^2} + x_i(r_d - r_f - \frac{1}{k})\frac{k}{2h}. \tag{22}$$

The discretization of the terminal condition (15) is given by

$$u_i^{N+1} = K(Kx_i - 1),\ i = 0, 1, \cdots, M+1.$$

Moreover discretization of boundary conditions (16)-(18), for $0 \le j \le N+1$, yields

$$u_0^j = 0,\ \frac{u_1^j - u_{-1}^j}{2h} = 0,\ \frac{u_{M+2}^j - u_M^j}{2h} = K^2. \tag{23}$$

Therefore we have

$$u_{-1}^j = u_1^j,\ u_{M+2}^j = 2K^2 h + u_M^j,\ j = 0, 1, \cdots, N+1. \tag{24}$$

Now, we want to find an explicit formula for determining the location of the stopping boundary. Substituting $i = 0$ into (19), one can get

$$u_0^j - \lambda_0^{j+1} S_f^j = \alpha_0 u_{-1}^{j+1} + \beta_0 u_0^{j+1} + \gamma_0 u_1^{j+1} - kq.$$

Plugging u_0^j and u_{-1}^{j+1} from (23) and (24), respectively, one can obtain

$$-\lambda_0^{j+1} S_f^j = (\alpha_0 + \gamma_0) u_1^{j+1} - kq. \tag{25}$$

On the other hand

$$\lambda_0^{j+1} = \frac{x_0}{2hS_f^{j+1}} (u_1^{j+1} - u_{-1}^{j+1}) = 0.$$

Substituting this relation into (25) yields

$$u_1^{j+1} = \frac{kq}{\alpha_0 + \gamma_0}, \ j = N, N-1, \cdots, 0. \tag{26}$$

Plugging $i = 1$ in (19), we have

$$u_1^j - \lambda_1^{j+1} S_f^j = \alpha_1 u_0^{j+1} + \beta_1 u_1^{j+1} + \gamma_1 u_2^{j+1} - kq.$$

Substituting u_0^j and u_1^{j+1} from (23) and (26), respectively, one can get

$$S_f^j = \frac{kq(1 - \beta_1 + \alpha_0\beta_0) + \gamma_1 u_2^{j+1}}{\lambda_1^{j+1}(\alpha_0 + \gamma_0)}, \ j = N, N-1, \cdots, 0, \tag{27}$$

where

$$\lambda_1^{j+1} = \frac{x_1}{2hS_f^{j+1}} (u_2^{j+1} - u_0^{j+1}) = \frac{x_1 u_2^{j+1}}{2hS_f^{j+1}}.$$

Note that, λ_1^{j+1} depends on the values computed from time step t_{j+1}. Solving the problem will be done backward, then when we are at time step t_j the values at time step t_{j+1} have been computed. Plugging $i = M + 1$ in (19), for $j = N, N-1, \cdots, 0$, we have

$$u_{M+1}^j - \lambda_{M+1}^{j+1} S_f^j = \alpha_{M+1} u_M^{j+1} + \beta_{M+1} u_{M+1}^{j+1} + \gamma_{M+1} u_{M+2}^{j+1} - kq.$$

Now, substituting u_{M+2}^{j+1} from (24), for $j = N, N-1, \cdots, 0$, yields

$$u_{M+1}^j - \lambda_{M+1}^{j+1} S_f^j = (\alpha_{M+1} + \gamma_{M+1}) u_M^{j+1} + \beta_{M+1} u_{M+1}^{j+1} + (2K^2 h \gamma_{M+1} - kq),$$

where

$$\lambda_{M+1}^{j+1} = \frac{x_{M+1}}{2hS_f^{j+1}} (u_{M+2}^{j+1} - u_M^{j+1}).$$

Using the fact that $x_{M+1} = x_\infty$ and substituting u_{M+2}^{j+1} from (24), one can obtain

$$\lambda_{M+1}^{j+1} = \frac{x_\infty K^2}{S_f^{j+1}}.$$

Now, to compute the price of European installment call option at (x_i, t_j) and to find the location of free or stopping boundary, we use (19) and (27), respectively. Therefore to compute the mentioned values, one can develop the following algorithm:

(1) Set $u_i^{N+1} = K(Kx_i - 1)$ for $0 \leq i \leq M + 1$.
(2) Compute $u_0^j = 0$ for $0 \leq j \leq N + 1$ and set $S_f^{N+1} = K$.
(3) Compute α_i, β_i and γ_i, for $0 \leq i \leq M + 1$, using (20)-(22).
(4) For $j = N, N-1, \cdots, 0$ compute:

- $$\lambda_1^{j+1} = \frac{x_1 u_2^{j+1}}{2hS_f^{j+1}},$$
$$S_f^j = \frac{kq(1-\beta_1+\alpha_0\beta_0)+\gamma_1 u_2^{j+1}}{\lambda_1^{j+1}(\alpha_0+\gamma_0)}.$$
- For $i = 2, 3, \cdots, M$ compute:
$$\lambda_i^{j+1} = \frac{x_i}{2hS_f^{j+1}}(u_{i+1}^{j+1} - u_{i-1}^{j+1}),$$
$$u_i^j - \lambda_i^{j+1} S_f^j = \alpha_i u_{i-1}^{j+1} + \beta_i u_i^{j+1} + \gamma_i u_{i+1}^{j+1} - kq.$$
- For $i = M+1$ compute:
$$\lambda_{M+1}^{j+1} = \frac{x_\infty K^2}{S_f^{j+1}},$$
$$u_{M+1}^j - \lambda_{M+1}^{j+1} S_f^j = (\alpha_{M+1}+\gamma_{M+1})u_M^{j+1} + \beta_{M+1}u_{M+1}^{j+1} + (2K^2 h\gamma_{M+1} - kq).$$

5. Numerical results

In this section, we want to implement the front fixing method using the algorithm presented in the previous section. As a result, the price of the European installment call option and the location of the stopping boundary will be computed. To do this, we first determine the parameters of the problem. Let's these parameters be given in table 1.

Table 1. Values of parameters

Parameter	Value
Maturity	$T = 0.25, 1$
Domestic risk-free interest rate	$r_d = 0.05$
Foreign risk-free interest rate	$r_f = 0.04$
Volatility	$\sigma = 0.2, 0.3$
Strike	$K = 100$

In this table, two levels are chosen for maturity, $T = 0.25, 1$, and for volatility, $\sigma = 0.2, 0.3$. Also, strike price is $K = 100$ and domestic and foreign risk-free interest rate are $r_d = 0.05, r_f = 0.04$, respectively.

Assume also that $x_\infty = 2$. We discretize the domain $[0.01, 2] \times [0, T]$ by step length $h = 0.1$ and time step $k = 0.005$. Running the mentioned algorithm, the prices of European installment call option are computed and these values are reported in table $2-4$.

In these tables, for underlying asset price, three values $S_0 \in \{95, 105, 115\}$ are chosen. Then, for each value of underlying asset, three values for installment rate $q \in \{1, 3, 6\}$ are given. In the next step, for different values of underlying asset and installment rate, the price of European installment call option is computed. As it is clear from table 2 to 4, an increase in values of installment rate causes a decrease in the prices of call options. On the other hand, when $q = 0$ European installment option becomes European vanilla option. Therefore, one can deduce that

TABLE 2. Installment call option prices for $\sigma = 0.2$ and $T = 0.25$

q	S_0	Price
1	95	0.5127
	105	5.0855
	115	11.6483
3	95	0.3114
	105	4.4297
	115	10.9595
6	95	0.5836
	105	3.8675
	115	9.3642

TABLE 3. Installment call option prices for $\sigma = 0.2$ and $T = 1$

q	S_0	Value
1	95	3.7039
	105	8.3959
	115	14.8574
3	95	2.2288
	105	6.6340
	115	12.9665
6	95	0.6772
	105	4.2746
	115	10.2556

TABLE 4. Installment call option prices for $\sigma = 0.3$ and $T = 0.25$

q	S_0	Price
1	95	2.3564
	105	7.0559
	115	13.5540
3	95	2.1363
	105	6.4596
	115	12.8486
6	95	2.1454
	105	5.7577
	115	11.2565

the premium of European vanilla call option is larger than the premium of European installment call option. But as a whole, the sum of the premium and the installments paid for European installment option is larger than the premium of the European vanilla option.

In figures $1-3$, stopping boundaries are depicted for $q = 6$, $q = 9$ and $q = 12$, respectively. It is seen that the stopping boundary is an increasing function of installment rate q. As it is clear from these figures, stopping boundary is not monotone with respect to time t but it is a

Table 5. Installment call option prices $\sigma = 0.3$ and $T = 1$

q	S_0	Price
1	95	7.4041
	105	12.1843
	115	18.6494
3	95	5.8452
	105	10.3411
	115	16.7585
6	95	6.1713
	105	4.2754
	115	13.9613

convex function of t. Moreover, stopping boundary curves corresponding to higher installment rate q are above the curves with lower installment rate. This proves that the termination of the contract is possible only for higher values of the underlying asset price when installment rate increases. In other word, since the stopping boundary is the critical asset price below which it is optimal to terminate the contract, then the least asset price to terminate the contract increases.

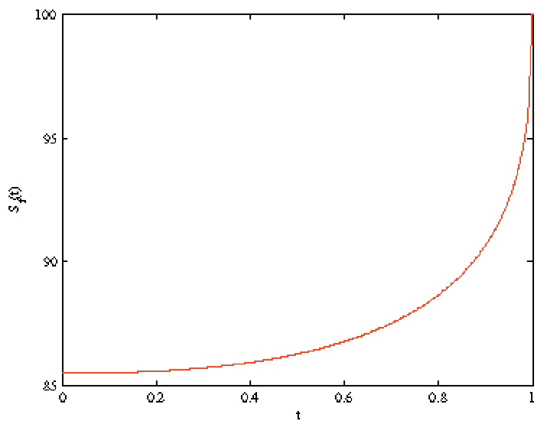

Figure 1. Stopping boundary $S_f(t)$ for $q = 6$

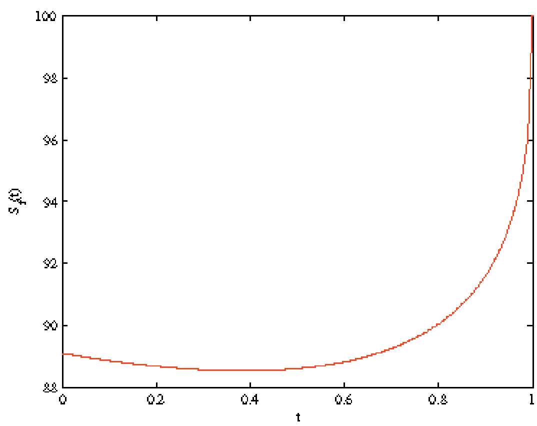

Figure 2. Stopping boundary $S_f(t)$ for $q = 9$

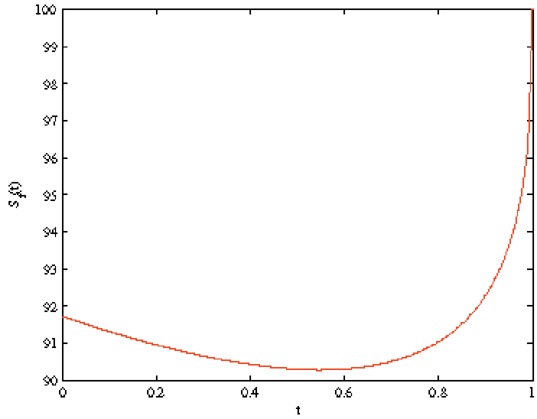

Figure 3. Stopping boundary $S_f(t)$ for $q = 12$

6. Conclusions

In this paper, the front fixing method is applied in combination with finite difference method to value European installment option. One of the advantages of front fixing is that it transforms the free or stopping boundary problem into a fixed boundary problem. Therefore, this technique is quite appropriate to price European installment call option which has a stopping boundary. Therefore, one can get the price of this type of option and its stopping boundary simultaneously without extra time. It is shown that the value of European continuous installment call option on foreign currency exchange rate has been obtained. Also, the graph of the stopping boundary was computed by front fixing method. One can conclude that this is an efficient method for valuing option problems with free boundary.

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Karim Ivaz is a Professor of Applied Mathematics at Faculty of Mathematical Sciences of University of Tabriz, Tabriz, Iran. His research interests are free boundary problem, numerical solution of PDE and IE, optimal control.

Ali Beiranvand is a Ph.D. candidate at Faculty of Mathematical Sciences of University of Tabriz, Tabriz, Iran. He received his master and bachelor degrees in Applied Mathematics. His fields of interests are financial mathematics, partial differential equations, financial engineering and risk management.